Problem with a solution proposed by Arkady Alt , San Jose , California, USA

Let $T_n(x)$ be the polynomial defined by recurrence $T_{n+1} = 2x \cdot T_n - T_{n-1},\ n \in \mathbb{N}$, and

initial conditions $T_0 = 1,\ T_1 = x$ (First Kind Chebishev's Polynomials).

Find all natural n for which holds inequality:

$$T_n(x) \geq (2^{n-2}+1)x^n - 2^{n-2}x^{n-1},\ \ x \in [1,\infty).$$

Solution.

We will prove that inequality $T_n(x) \geq (2^{n-2}+1)x^n - 2^{n-2}x^{n-1},\ \ x \in [1,\infty)$ holds only for

$n \in \{2,3,4,5,6,7\}$.

Denote $R_n(x) := T_n(x) - (2^{n-2}+1)x^n + 2^{n-2}x^{n-1}$. We have $R_n(1) = 0$ for all natural n because $T_n(1) = 1, n \in \mathbb{N}$. Then $R_n(x)$ is divisible by $x-1$ and that determine for any $n \in \mathbb{N}$ polynomial-quatient $Q_n(x) = \dfrac{R_n(x)}{x-1}$. Thus our problem becomes:

Find all natural n such that $Q_n(x) \geq 0$ for all $x \in [1,\infty)$.

Substitution $T_n(x) = R_n(x) + (2^{n-2}+1)x^n - 2^{n-2}x^{n-1}$ in the recurrence for $T_n(x)$, give us recurrence for $R_n(x)$:

$R_{n+1} - 2xR_n + R_{n-1} = T_{n+1} - (2^{n-2}+1)x^{n+1} + 2^{n-1}x^n - 2x\,T_n + (2^{n-1}+2)x^{n+1} - 2^{n-1}x^n + T_{n-1} - (2^{n-3}+1)x^{n-1} + 2^{n-3}x^{n-2} = x^{n+1} - (2^{n-3}+1)x^{n-1} + 2^{n-3}x^{n-2} = x^{n-2}(x^3 - (2^{n-3}+1)x + 2^{n-3}) = x^{n-2}(x-1)(x^2+x-2^{n-3})$ and $R_1(x) = \dfrac{1-x}{2},\ R_2(x) = 2x^2 - 1 - 2x^2 + x = x-1.$

Hence, we have for $Q_n(x)$ recurrence

(1) $Q_{n+1}(x) = 2xQ_n(x) - Q_{n-1}(x) + x^{n-2}(x^2+x-2^{n-3}), n \geq 2$

with initial conditions $Q_1(x) = -\dfrac{1}{2}, Q_2(x) = 1.$

Using recurrence (1), we obtain that for all $x \in [1,\infty)$:

$Q_3(x) = x(x+3) > 0,\ Q_4(x) = 3x^3 + 7x^2 - x - 1 > 0,\ Q_5(x) = 7x^4 + 16x^3 - 4x^2 - 5x > 0,$
$Q_6(x) = 15x^5 + 33x^4 - 12x^3 - 10x^2 > 0, Q_7(x) = 31x^6 + 67x^5 - 39x^4 - 36x^3 + 4x^2 + 5x > 0.$

So, obtain that for $n \in \{2,3,4,5,6,7\}$ and $x \in [1,\infty)$ holds $Q_n(x) \geq 0.$

Denoting $q_n := Q_n(1),$ we get recurrence for q_n :

(2) $q_{n+1} - 2\,q_n + q_{n-1} = 2 - 2^{n-3},\ n \geq 2$

and $q_1 = -\dfrac{1}{2},\ q_2 = 1.$ Solving this recurrence we obtain $q_n = n(n-1) - 2^{n-2}$ for $n \in \mathbb{N}.$

From that follow $q_n < 0$ for all $n \geq 8.$

($q_3 = 4, q_4 = 8, q_5 = 12, q_6 = 14, q_7 = 10, q_8 = -8$ and $q_n < 0$ for all $n \geq 8$ by Math. Induction .

From sign preservation property of polynomials follow that there is some $\varepsilon > 0$ such that

$Q(x) < 0$ for all $t \in (1,\varepsilon)$).